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## Liquid Crystals

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# PRELIMINARY COMMUNICATIONS 

# Eigenvalues of propagation through liquid crystals 

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#### Abstract

Analytical expressions are given for the propagation constant through an homogeneous liquid crystal, following the $4 \times 4$ matrix formulation of Berreman. Also propagation through polarizers can be calculated analytically; it is a special case of a situation described by Teitler and Henvis.


Light propagation through a liquid crystal is complicated because the medium is anisotropic and inhomogeneous. If the $z$ axis is chosen to be perpendicular to the glass plates, the $x$ axis can always be chosen such that the propagation is in the $x z$ plane (see figure 1). The director orientation of the liquid crystal is described by a twist angle $\vartheta$ and a tilt angle $\varphi$. Normally $\vartheta$ and $\varphi$ are functions of $z$. The optical dielectric constant along the director $\varepsilon_{1}$ differs from the dielectric constant perpendicular to the director, $\varepsilon_{2}$. We can look for solutions to the wave equation of the form

$$
E=E(z) \exp \left(j\left(\omega t-k_{x} x\right)\right)
$$

and

$$
H=H(z) \exp \left(j\left(\omega t-k_{x} x\right)\right)
$$

where $k_{x}=k_{0} \sin \alpha, k_{0}=\omega / c=\omega \sqrt{\varepsilon_{0}} \mu_{0}$. The angle $\alpha$ is the angle of incidence in free space. Berreman [1] analyses this problem in terms of the components $E_{x}, Z_{0} H_{y}, E_{y}$, $-Z_{0} H_{x}$ where $Z_{0}=\sqrt{ } \mu_{0} / \varepsilon_{0}$. Finally the liquid crystal is divided into a sufficiently large number of slabs, in which $\vartheta$ and $\varphi$ are considered constant. In such a homogeneous slab, we can look for eigenmodes for which the $z$ dependence is of the form $\exp \left(-j m k_{0} z\right)$. This leads to a fourth order equation for $m$

$$
\left|\begin{array}{cccc}
\Delta_{11}-m & \Delta_{12} & \Delta_{13} & 0  \tag{1}\\
\Delta_{21} & \Delta_{11}-m & \Delta_{23} & 0 \\
0 & 0 & -m & 1 \\
\Delta_{23} & \Delta_{13} & \Delta_{43} & -m
\end{array}\right|=0
$$

where the $\Delta_{i j}$ are given in equation (5) of [1] ( $\varepsilon_{a}=\varepsilon_{b}=\varepsilon_{2}, \varepsilon_{c}=\varepsilon_{1}, \varphi \rightarrow \vartheta+\pi / 2$, $\vartheta \rightarrow \pi / 2-\vartheta, \psi=0, x=\sin \alpha$ ). Although this equation for $m$ can be solved with the aid of a computer, it is much faster if analytical expressions for $m$ are known. These can be found based on the fact that there is cylindrical symmetry around the director $\mathbf{n}$. Indeed the two modes, for which the polarization direction


Figure 1. The twist angle 9 and the tilt angle $\varphi$.


Figure 2. The $\varepsilon_{2}$ modes.
(direction of $E$ ) is perpendicular to the director and the propagation direction $k$, see only the dielectric constant $\varepsilon_{2}$, such that $k^{2}=k_{0}^{2} \varepsilon_{2}$ and so

$$
\begin{equation*}
m^{2}=\varepsilon_{2}-\sin ^{2} \alpha . \tag{2}
\end{equation*}
$$

Equation (1) is therefore of the form

$$
\left(-m^{2}+\varepsilon_{2}-\sin ^{2} \alpha\right)\left(m^{2}+a m-b\right)=0
$$

with $a$ equal to the coefficient of $m^{3}$ which is $2 \Delta_{11}$ and $b\left(\varepsilon_{2}-\sin ^{2} \alpha\right)$ as the coefficient of $m^{0}$

$$
\left|\begin{array}{lll}
\Delta_{11} & \Delta_{12} & \Delta_{13} \\
\Delta_{21} & \Delta_{11} & \Delta_{23} \\
\Delta_{23} & \Delta_{13} & \Delta_{43}
\end{array}\right| .
$$

Rather lengthy but straightforward calculations show that

$$
\begin{aligned}
& a=2 \sin \alpha \frac{\Delta \varepsilon \cos \varphi \sin \varphi \cos \vartheta}{\varepsilon_{2}+\Delta \varepsilon \sin ^{2} \varphi} \\
& b=\frac{\varepsilon_{2}\left(\varepsilon_{1}-\sin ^{2} \alpha\right)-\sin ^{2} \alpha \Delta \varepsilon \cos ^{2} \varphi \cos ^{2} \vartheta}{\varepsilon_{2}+\Delta \varepsilon \sin ^{2} \varphi}
\end{aligned}
$$

and thus

$$
\begin{equation*}
m=-\frac{a}{2} \pm \frac{1}{2}\left(a^{2}+4 b\right)^{1 / 2}, \quad \Delta \varepsilon=\varepsilon_{1}-\varepsilon_{2} \tag{3}
\end{equation*}
$$

As soon as these eigenvalues are known, a computer must be used to calculate the eigenmodes, and proceed to calculate the transmission coefficient as explained in [1].

For the calculation of the transmission through a liquid crystal display, we must consider the transmission through a polarizer, through glass, through the liquid crystal, again through glass, and finally through a second polarizer. This method can also be used for the transmission through the polarizers. These have also a director with cylindrical symmetry, but which is oriented in the $x y$ plane, and therefore $\varphi$ is zero. The dielectric tensor $\varepsilon$ is complex, since $\varepsilon_{1}$ and $\varepsilon_{2}$ contain loss factors. For positive dichroism $\varepsilon_{1}$ has the larger loss factor (such that the polarization direction $\vartheta_{\mathrm{p}}$ is perpendicular to the director in the xy plane; $\vartheta_{\mathrm{p}}=\vartheta+\pi / 2$ ), for negative dichroism $\varepsilon_{2}$ has the larger loss factor and $\vartheta_{\mathrm{p}}=\vartheta$. In the special case that $\varphi=0$, equations (2) and (3) reduce to

$$
\begin{align*}
& m^{2}=\varepsilon_{2}-\sin ^{2} \alpha  \tag{4}\\
& m^{2}=b=\varepsilon_{1}-\sin ^{2} \alpha\left(1+\frac{\Delta \varepsilon}{\varepsilon_{2}} \cos ^{2} \vartheta\right) \tag{5}
\end{align*}
$$

In a more general case, when there is no cylindrical symmetry around the director, the director $\left(\varepsilon_{1}\right)$ and one principal axis $\left(\varepsilon_{2}\right)$ are oriented in the $x y$ plane, the third principal axis $\left(\varepsilon_{3}\right)$ is along the $z$ axis. The eigenvalues are again given by equation (1) where the $\Delta_{i j}$ can be found in equation (5) of [1] ( $\varepsilon_{a}=\varepsilon_{1}, \varepsilon_{b}=\varepsilon_{2}, \varepsilon_{c}=\varepsilon_{3}, \vartheta=0, \varphi \rightarrow \vartheta$, $\psi=0, x=\sin \alpha$ ). This gives in our notation, with $\Delta \varepsilon=\varepsilon_{1}-\varepsilon_{2}$

$$
\left.\begin{array}{l}
\Delta_{12}=1-\frac{\sin ^{2} \alpha}{\varepsilon_{3}}, \quad \Delta_{21}=\varepsilon_{1} \cos ^{2} \vartheta+\varepsilon_{2} \sin ^{2} \vartheta  \tag{6}\\
\Delta_{3}=\Delta \varepsilon \sin \vartheta \cos \vartheta, \quad \Delta_{43}=\varepsilon_{1} \sin ^{2} \vartheta+\varepsilon_{2} \cos ^{2} \vartheta-\sin ^{2} \alpha .
\end{array}\right\}
$$

All other elements are zero. This leads to the equation

$$
\begin{equation*}
m^{4}-m^{2}\left(\Delta_{12} \Delta_{21}+\Delta_{43}\right)+\Delta_{12} \Delta_{21} \Delta_{43}-\Delta_{23} \Delta_{12} \Delta_{23}=0, \tag{7}
\end{equation*}
$$

or

$$
\begin{gathered}
m^{4}-a m^{2}+b=0 \\
a=\varepsilon_{1}+\varepsilon_{2}-\frac{\sin ^{2} \alpha}{\varepsilon_{3}}\left(\varepsilon_{3}+\varepsilon_{1} \cos ^{2} \vartheta+\varepsilon_{2} \sin ^{2} \vartheta\right) \\
b=\left[\varepsilon_{1} \varepsilon_{2}-\sin ^{2} \alpha\left(\varepsilon_{1} \cos ^{2} \vartheta+\varepsilon_{2} \sin ^{2} \vartheta\right)\right]\left(1-\frac{\sin ^{2} \alpha}{\varepsilon_{3}}\right)
\end{gathered}
$$

Again this equation can easily be solved analytically. Teitler and Henvis [2] already found this result for an asymmetric tensor. We can easily verify that for $\varepsilon_{3}=\varepsilon_{2}$, equation (7) leads to equations (4) and (5).

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## References

[1] Berreman, D. W., 1973, J. opt. Soc. Am., 63, 1374.
[2] Teitler, S., and Henvis, B. W., 1970, J. opt. Soc. Am., 60, 830.

